Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $f(x) = \begin{cases} x & if\ x \epsilon Q \\ 0 & if\ x \epsilon Q^c \end{cases}$

Prove that f is continuous only at $a = 0$

Proof:

Let $\epsilon > 0$ be given

Let us take $\delta = \epsilon$

For every $x \in \mathbb{R}$ with $|x - 0| < \delta$

$|f(x) - f(0)| = |f(x)| \leq |x| < \delta = \epsilon$

$\therefore f$ is continuous at $a = 0$.

Let $a \neq 0 \Longrightarrow |a| > 0$

Suppose f is continuous at a

Let $\epsilon = |a|$ clearly $\epsilon > 0$

$\Longrightarrow \exists\ \delta > 0$ such that $x\ \epsilon\ \mathbb{R}, |x - a| < \delta \ \Longrightarrow\ |f(x) - f(a)| < |a| \quad \cdots\cdots (1)$

Let $a \in Q$

Let us choose $x \in Q^c$ such that $|x - a| < \delta$

$\Longrightarrow\ |f(x) - f(a)| < |a|$ (By (1))

$\Longrightarrow\ |0 - a| < |a|$

$i.e., |a| < |a|$

Which is $\Longrightarrow\Longleftarrow$

Let $a \in Q^c$ and $a > 0$

Let us choose $x \in Q$ such that $a < x < a + \delta$

$\Longrightarrow |x - a| < \delta$

$\Longrightarrow\ |f(x) - f(a)| < |a|$ (By (1))

$\Longrightarrow\ |x - 0| < |a|$

$i.e., |x| < |a|$

Which is $\Longrightarrow\Longleftarrow$ that $0 < a < x$

Let $a \in Q^c$ and $a < 0$

Let us choose $x \in Q$ such that $a - \delta < x < a$

$\Longrightarrow |x - a| < \delta$ and $0 < |a| < |x|$

$\Longrightarrow |f(x) - f(a)| < |a|$ (By (1))

$\Longrightarrow |x - 0| < |a|$

$i.e., |x| < |a|$

Which is $\Longrightarrow\Longleftarrow$ that $0 < |a| < |x|$

$\therefore f$ is not continuous at any $a \neq 0$.

Hence proved.